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Subjoinings of affine Kac–Moody algebras*

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Abstract. Subjoining among affine Kac-Moody algebras is a relation generalising the algebra-subalgebra relation. It is analogous to subjoinings among semisimple finite Lie algebras. A list of maximal subjoinings of equal rank affine Kac-Moody algebras is presented.. It is conjectured that there are no other maximal subjoinings.

Résumé. La sous-jonction parmi les algèbres de Kac-Moody affine est une relation généralisant la relation algèbre-sous-algèbre. Elle est analogue aux sous-jonctions parmi les algèbres de Lie semi-simples finies. Une liste des sous-jonctions maximales pour des algèbres de Kac-Moody affines est présentée. On avance la conjecture qu'il n'y a pas d'autres sous-jonctions maximales.

1. Introduction

Subjoinings among semisimple finite-dimensional Lie algebras have been studied recently [1-5] with extensive examples shown in [6]. Subjoining generalises the familiar inclusion relation of semisimple Lie algebras in semisimple Lie algebras in that the latter becomes a special case. Here we initiate the study of the analogous relation among affine Kac-Moody algebras by demonstrating many examples of it. In fact, we believe that we list here all maximal proper subjoinings of affine Kac-Moody algebras, but we do not have a proof of that assertion.

Inclusions, or equivalently, embeddings, of Lie algebras have been extensively exploited for a long time in mathematics and in its applications. It is natural to expect that the generalisation should also prove useful.

The most striking and perhaps the most revealing feature of subjoining is the possibiliy of mapping irreducible representations of a simple Lie algebra which admit a weight decomposition, in particular all the highest weight representations, to linear combinations of irreducible representations of another Lie algebra. The coefficients of such linear combinations (called 'multiplicities') are integers, not necessarily positive ones. The mapping is called a *branching rule.* Let $A, B, C, D, \ldots$ denote semisimple Lie algebras or affine Kac-Moody algebras. Two subjoinings of B to A are (linearly) equivalent if their branching rules coincide at least for one faithful representation of

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A because then they necessarily coincide for all of them. A subjoining which is not an embedding is called a *proper subjoining*. An embedding of the algebra B in A is denoted a usual by $A \supset B$ while a proper subjoining of B to A is written $A > B$. In the former case B is a subalgebra of A, in the latter one we say that B is properly subjoined to A, or B is a hypoalgebra of A; both cases exemplify a subjoining of B to A.

Embeddings and proper subjoinings can be decomposed into chains,

$$A > B \supset C > \ldots \supset D$$

where the outcome of several successive embeddings and proper subjoinings is either an embedding, $A \supset D$, or a proper subjoining, $A > D$. A proper subjoining $A > C$ is not maximal if it can be decomposed into one of the chains

$$A > B > C \qquad A > B \supset C \qquad A \supset B > C.$$

Another curious feature of subjoining is the existence of infinitely many non-equivalent subjoinings among isomorphic Lie algebras. In particular, there exist chains of subjoinings

$$A > A > A > \ldots > A$$

of arbitrary length with all links nontrivial.

In general a maximal reductive subalgebra of a given simple Lie algebra may not be maximal among subjoinings. That observation was the motivation for the invention of the subjoining [1]. Originally, the subjoining was used as a device in computing certain generating functions in representation theory for $A \supset C$ in terms of generating functions for $A > B$ and $B > C$.

The maximal embeddings among semisimple Lie algebras of equal rank were classified by Borel and de Siebenthal [7], maximal embeddings among all semisimple Lie algebras were determined by Dynkin [8] and maximal subjoinings of equal rank reductive subalgebras in simple Lie algebras were classified by Moody and Pianzola [5]. There are no examples known of maximal proper subjoinings which are not of the equal rank type.

We use the names of affine Kac-Moody algebras identified, for example, in table Aff1 and Aff2 of [9] or tables 1 and 2 of [10]. For the properties of semisimple Lie algebras, see [11].

A round bracket denotes an irreducible representation, specified by its highest weight; a square bracket is any weight of a representation; a curly bracket contains a dominant weight and denotes the corresponding Weyl orbit. The relative position of a weight in an irreducible weight system is given by its depth, which is shown as a subscript whenever needed (see the definition following (2.2)).

2. Proper maximal subjoinings of affine Kac–Moody algebras

In [2] a semisimple algebra H or rank l_H is said to be subjoined to a semisimple algebra G of rank $l_G \geqslant l_H$ (the relationship is denoted $G > H$) if there exists an $l_G \times l_H$ projection matrix P of rank l_H such that there is a branching rule for $G > H$

$$P\Phi(G) = \Psi(H) - \Omega(H) \tag{2.1}$$

Here $\Phi(G)$ is the weight system of a representation of G, and $\Psi(H)$ and $\Omega(H)$ are weight systems of representations of H. The multiplicity, always a non-negative integer, of a weight in $\Psi(H)-\Omega(H)$ is its multiplicity in $\Psi(H)$ less its multiplicity in $\Omega(H)$. In the special case that $\Omega(H)$ is the empty set for every $\Phi(G)$, H is a subalgebra of G.

A projection matrix P acts on individual weights of the system $\Phi(G)$, transforming them into weights of the difference $\Psi(H)-\Omega(H)$. For many cases of interest the projection matrices can be found in [10] and [12]. Here we adopt an equivalent definition (and easier to visualise and work with, especially when G is an affine Kac-Moody algebra) that H is subjoined to G if the matrix P above preserves Weyl group symmetry of the weight system, i.e., if P operating on a set of weights invariant under the Weyl group of G yields a set invariant under the Weyl group of H. It can be shown that this is equivalent to the requirement that the Weyl group $W(H)$ is a subgroup of the Weyl group $W(G)$ of G. The existence of a branching rule (2.1) then follows.

To begin, we now restrict our attention to subjoinings of the type $G>G$ between two isomorphic algebras.

First we dispose of some trivial examples of subjoinings (actually embeddings) for which the projection matrix P is an $l_G \times l_G$ permutation matrix. Since we write weight components in a fundamental weights basis, it is convenient to think of the action of P on the nodes of the Dynkin diagram of G. Examples of trivial embeddings, as defined above, include, for the Kac-Moody algebras $A_l^{(1)}$, cyclical renumbering of the nodes, or reversal of the direction of numbering; for $C_l^{(1)}$ and $D_{l+1}^{(2)}$, $l \geq 2$, and $E_7^{(1)}$, there is reflection of the Dynkin diagram in its centre; for $D_l^{(1)}$, $l \geq 4$, there is reflection in the centre and also exchange of the two nodes at one end; for $E_6^{(1)}$ there are permutations of the three 'tails' emanating from the centre node; finally, for $B_l^{(1)}$ and $A_{2l-1}^{(2)}$, $l \geq 3$, there is an interchange of the two nodes at the left end. Without the exclusion of these trivial cases, no subjoining or embedding would be maximal.

Our first example of a non-trivial subjoining $G>G$ is a change of scale; but for Kac-Moody algebras, since there are no Weyl reflections along the imaginary root δ, we are permitted to change the scale independently in the real and imaginary directions. Since the projection matrix P is diagonal we write its elements in a row with no ambiguity. For a change of scale in real directions the projection matrix is

$$P = \mathrm{diag}(k, k, \ldots, k \mid 1) \qquad k > 1 \tag{2.2}$$

where k is a integer. We have added an lth component to P (the others are numbered $0, 1, \ldots, l-1$) which acts on the depth of a weight (the negative of its imaginary component). That the subjoining (2.2) is maximal is equivalent to the requirement that k is prime, except when

$$k = 2 \text{ and } G \text{ is } F_4^{(1)},\ E_6^{(2)},\ C_l^{(1)},\ D_{l+1}^{(2)},\ A_{2l}^{(2)},\ B_l^{(1)},\ A_{2l-1}^{(2)},$$

or

$$k = 3 \text{ and } G \text{ is } G_2^{(1)} \text{ or } D_4^{(3)}.$$

In these cases, the subjoinings are non-maximal because they are products of subjoinings of the types to be described in the following five paragraphs. For the purpose of effecting Weyl reflections after the subjoining (2.2), the Cartan matrix should be modified from

$$\left(\begin{array}{c|c} C & \begin{matrix} -1 \\ 0 \\ \vdots \\ 0 \end{matrix} \end{array}\right) \quad \text{to become} \quad \left(\begin{array}{c|c} C & \begin{matrix} -k^{-1} \\ 0 \\ \vdots \\ 0 \end{matrix} \end{array}\right).$$

The elements of the last column, which we have added to C, represent the negatives of the coefficients of δ in the simple roots $\alpha_0, \alpha_1, \ldots, \alpha_{l-1}$. For a change of scale in the imaginary direction, the projection matrix is

$$P = \mathrm{diag}(1, 1, \ldots, 1 \,|\, k) \tag{2.3}$$

and the Cartan matrix, for effecting Weyl reflections after the subjoining, is

$$\left(\begin{array}{c|c} & -k \\ & 0 \\ C & \vdots \\ & 0 \end{array}\right).$$

As in the finite-algebra case we can define a subjoining in both directions between a pair of dual algebras (two algebras are said to be dual if their Dynkin diagrams are interchanged by interchanging long and short roots in a connected Dynkin diagram); the subjoining involves an increase of the labels corresponding to the longer roots by a factor equal to the square of the ratio of the lengths of the roots. A similar subjoining is possible between any algebras whose Dynkin diagrams have the same shape (number of nodes the same and angles between pairs of corresponding simple roots the same). Thus for $B_l^{(1)} > A_{2l-1}^{(2)} (l \geqslant 3)$ we have

$$P = \mathrm{diag}(2, 2, \ldots, 2, 1 \,|\, 1) \tag{2.4}$$

and for $A_{2l-1}^{(2)} > B_l^{(1)}$

$$P = \mathrm{diag}(1, 1, \ldots, 1, 2 \,|\, 1). \tag{2.5}$$

For $F_4^{(1)} > E_6^{(2)}$ the projection matrix is

$$P = \mathrm{diag}(2, 2, 2, 1, 1 \,|\, 1) \tag{2.6}$$

and for $E_6^{(2)} > F_4^{(1)}$

$$P = \mathrm{diag}(1, 1, 1, 2, 2 \,|\, 1). \tag{2.7}$$

For $G_2^{(1)} > D_4^{(3)}$ we have

$$P = \mathrm{diag}(3, 3, 1 \,|\, 1) \tag{2.8}$$

and for $D_4^{(3)} > G_2^{(1)}$

$$P = \mathrm{diag}(1, 1, 3 \,|\, 1). \tag{2.9}$$

The three algebras $C_l^{(1)}$, $A_{2l}^{(2)}$ and $D_{l+1}^{(2)}$, $l \geqslant 2$, are related cyclically by maximal subjoinings.

For $C_l^{(1)} > A_{2l}^{(2)}$ we have

$$P = \mathrm{diag}(2, 1, \ldots, 1 \,|\, 1) \tag{2.10}$$

for $A_{2l}^{(2)} > D_{l+1}^{(2)}$ we have

$$P = \mathrm{diag}(1, 1, \ldots, 1, 2 \,|\, 1) \tag{2.11}$$

and for $D_{l+1}^{(2)} > C_l^{(1)}$

$$P = \mathrm{diag}(1, 2, \ldots, 2, 1 \,|\, 1). \tag{2.12}$$

For $A_1^{(1)} > A_2^{(2)}$ we have

$$P = \mathrm{diag}(2, 1 \,|\, 1) \tag{2.13}$$

and for $A_2^{(2)} > A_1^{(1)}$

$$P = \mathrm{diag}(1, 2 | 1). \tag{2.14}$$

For the subjoinings (2.4)-(2.14) the Cartan matrix for effecting Weyl reflections after the subjoining is

$$\left(\begin{array}{c|c} & -k^{-1} \\ & 0 \\ C' & \vdots \\ & 0 \end{array}\right)$$

where C' is the usual Cartan matrix for the subjoined algebra and k is the zeroth element of the projection matrix P.

Before concluding this section we show, as an example, that the projection matrix $P = \mathrm{diag}(3, 3, 1 | 1)$ for the subjoining $G_2^{(1)} > D_4^{(3)}$ does actually preserve Weyl symmetry. The Cartan matrices (negative sign is written as an overbar on a number),

$$\begin{pmatrix} 2 & \bar{1} & 0 & \bar{1} \\ \bar{1} & 2 & \bar{3} & 0 \\ 0 & \bar{1} & 2 & 0 \end{pmatrix} \qquad \text{for } G_2^{(1)}$$

and

$$\begin{pmatrix} 2 & \bar{1} & 0 & \overline{3^{-1}} \\ \bar{1} & 2 & \bar{1} & 0 \\ 0 & \bar{3} & 2 & 0 \end{pmatrix} \qquad \text{for } D_4^{(3)}$$

can be used to effect Weyl reflections in the two weight systems: to reflect along the direction of the simple root α_i a weight whose ith component is λ_i, subtract λ_i times the ith row of the Cartan matrix. In table 1 we reverse separately each component of the $G_2^{(1)}$ weight $[\lambda_0, \lambda_1, \lambda_2]_d$, and in parallel each component of the $D_4^{(3)}$ weight $[3\lambda_0, 3\lambda_1, \lambda_2]_d$ obtained from it by the projection matrix P. For a general proof (G is the algebra and H the subjoined algebra) we need to consider only one pair of weight components, say $[\ldots, \lambda_i, \ldots, \lambda_j, \ldots]_d$ of G which for simplicity we write as $[\lambda_i, \lambda_j]_d$. The G Cartan submatrix corresponding to the subdiagram with n lines joining the longer root α_i to the shorter root α_j is

$$C_G = \begin{pmatrix} 2 & \bar{n} & -\delta_{i0} \\ \bar{1} & 2 & -\delta_{j0} \end{pmatrix}$$

while that to be used for H after the projection is

$$C_H = \left(\begin{array}{cc|c} 2 & \bar{1} & -n^{-1}\delta_{i0} \\ \bar{n} & 2 & -\delta_{j0} \end{array}\right).$$

The projection submatrix is $\mathrm{diag}(n, 1 | 1)$. Table 2 shows the result of reversing each component, and, in parallel, each component of the projected weight of the subjoined algebra.

We conjecture without proof that we have given a complete list of maximal equal rank subjoinings of affine Kac-Moody algebras.

Table 1. Elementary Weyl reflections of an arbitrary $G_2^{(1)}$ weight λ and of the corresponding $D_4^{(3)}$ weight.

	$G_2^{(1)}$ weights	$D_4^{(3)}$ weights
λ	$[\lambda_0, \lambda_1, \lambda_2]_d$	$[3\lambda_0, 3\lambda_1, \lambda_2]_d$
$r_0\lambda$	$[-\lambda_0, \lambda_0+\lambda_1, \lambda_2]_{d+\lambda_0}$	$[-3\lambda_0, 3\lambda_0+3\lambda_1, \lambda_2]_{d+\lambda_0}$
$r_1\lambda$	$[\lambda_0+\lambda_1, -\lambda_1, 3\lambda_1+\lambda_2]_d$	$[3\lambda_0+3\lambda_1, -3\lambda_1, 3\lambda_1+\lambda_2]_d$
$r_2\lambda$	$[\lambda_0, \lambda_1+\lambda_2, -\lambda_2]_d$	$[3\lambda_0, 3\lambda_1+3\lambda_2, -\lambda_2]_d$

Table 2. Elementary Weyl reflections of the ith and jth components of an arbitrary G weight λ and of the corresponding weight of the algebra H subjoined to G; n lines join the nodes in the Dynkin diagram of G corresponding to a longer root α_i and a shorter root α_j.

	G	H
λ	$[\lambda_i, \lambda_j]_d$	$[n\lambda_i, \lambda_j]_d$
$r_i\lambda$	$[-\lambda_i, n\lambda_i+\lambda_j]_{d+\delta_{i0}\lambda_i}$	$[-n\lambda_i, n\lambda_i+\lambda_j]_{d+\delta_{i0}\lambda_i}$
$r_j\lambda$	$[\lambda_i+\lambda_j, -\lambda_j]_{d+\delta_{j0}\lambda_j}$	$[n\lambda_i+n\lambda_j, -\lambda_j]_{d+\delta_{i0}\lambda_i}$

3. Orbit–orbit branching rules

Branching rules for representations of an algebra G to subalgebra, or to subjoined algebra H, can be found by a three-step process. The steps are decomposition of

(i) a weight system of an irreducible representation $\Phi(G)$ to Weyl orbits (G-orbits) of G

(ii) G-orbits to H-orbits, and

(iii) H-orbits to an integer linear combination of weight systems of irreducible representations of H.

Step (i) amounts to determining the orbit multiplicities in an irreducible representation of G; efficient methods for implementing it are found in [10] and [14].

Here we solve the problem of step (ii). For the subjoinings described in section 2 they are all so simple as to be trivial. Our projection matrices have only non-negative elements, so they carry a dominant (non-dominant) weight of the algebra into a dominant (non-dominant) weight of the subjoined algebra. Thus each algebra orbit branches to a single subjoined algebra orbit, with orbit labels related by the projection matrix.

Orbit multiplicities of suitably ordered representations form a triangular matrix [10]; step (iii) can be achieved by inverting this matrix, or directly, as in [14].

4. Examples

We consider four examples of branching rules, making use of the orbit multiplicities from tables of [10] or [14].

Example 1. $A_1^{(1)} > A_1^{(1)}$, $P = \mathrm{diag}(2, 2|1)$
Let us take the irreducible representation of $A_1^{(1)}$ with the highest weight (1 0) and draw the first few of its weights as they appear by successive subtractions of the simple roots of the Lie algebra:

$$\begin{array}{ccc} & \boxed{1\,0}_0 & \\ \boxed{\bar{1}\,2}_1 & & \\ & \boxed{1\,0}_1 & \\ \boxed{\bar{1}\,2}_2 & & \boxed{3\,\bar{2}}_1 \\ & 2\boxed{1\,0}_2 & \\ \vdots & \vdots & \vdots \end{array} \tag{4.1}$$

A subscript at a weight-box indicates the depth; the multiplicity of a weight is shown in front of the box whenever it is >1. The significant information here is only the multiplicities of the repeated dominant weight $\{1\ 0\}$ and its depths. Thus it suffices to draw only

$$\begin{array}{c} \boxed{1\,0}_0 \\ \boxed{1\,0}_1 \\ 2\boxed{1\,0}_2 \\ \vdots \end{array} \tag{4.2}$$

from the weight system. Applied to a weight $[\lambda_0\,\lambda_1]_d$ the projection matrix P gives

$$P[\lambda_0\,\lambda_1]_d = [2\lambda_0\,2\lambda_1]_d. \tag{4.3}$$

Hence (4.2) projects as

$$P \quad \begin{array}{c} \boxed{1\,0}_0 \\ \boxed{1\,0}_1 \\ 2\boxed{1\,0}_2 \\ \vdots \end{array} \quad \longrightarrow \quad \begin{array}{c} \boxed{2\,0}_0 \\ \boxed{2\,0}_2 \\ 2\boxed{2\,0}_4 \\ \vdots \end{array} \tag{4.4}$$

In order to decompose the result into contributions of irreducible representations, we write

$$\begin{array}{c} \boxed{2\,0}_0 \\ \\ \boxed{2\,0}_1 \\ \\ 2\boxed{2\,0}_2 \\ \vdots \end{array} \;=\; \begin{array}{cccccc} \boxed{2\,0}_0 & & & & & \\ \boxed{0\,2}_1 & -\boxed{0\,2}_1 & & & & \\ \boxed{2\,0}_1 & -\boxed{2\,0}_1 & +\boxed{2\,0}_1 & & & \\ 2\boxed{0\,2}_2 & -\boxed{0\,2}_2 & +\boxed{0\,2}_2 & -2\boxed{0\,2}_2 & & \\ 3\boxed{2\,0}_2 & -2\boxed{2\,0}_2 & +\boxed{2\,0}_2 & -2\boxed{2\,0}_2 & +2\boxed{2\,0}_2 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{array} \tag{4.5}$$

where on the right-hand side we have vertically aligned dominant weigh from the same irreducible representation. We then write the branching rule (4.5) in concise form using only highest weights:

$$(1\,0)_0 > (2\,0)_0 - (0\,2)_1 + (2\,0)_1 - 2(0\,2)_2 + 2(2\,0)_2 + \ldots \tag{4.6}$$

where the subscripts indicate the relative vertical displacement in (4.5).

Example 2. $A_1^{(1)} > A_1^{(1)}$, $P = \mathrm{diag}(1, 1|2)$.
Again we consider the irreducible representation (1 0) and write

$$P \quad \begin{array}{r} \boxed{1\,0}_0 \\ \boxed{1\,0}_1 \\ 2\boxed{1\,0}_2 \\ \vdots \end{array} \quad \longrightarrow \quad \begin{array}{r} \boxed{1\,0}_0 \\ \boxed{1\,0}_2 \\ 2\boxed{1\,0}_4 \\ \vdots \end{array}. \tag{4.7}$$

Following the format of (4.5) we have

$$\begin{array}{r} \boxed{1\,0}_0 \\ \\ \boxed{1\,0}_2 \\ \\ 2\boxed{1\,0}_4 \\ \vdots \end{array} \; = \; \begin{array}{rrrr} \boxed{1\,0}_0 & & & \\ \boxed{1\,0}_1 & -\boxed{1\,0}_1 & & \\ 2\boxed{1\,0}_2 & -\boxed{1\,0}_2 & & \\ 3\boxed{1\,0}_3 & -2\boxed{1\,0}_3 & -\boxed{1\,0}_3 & + \\ 5\boxed{1\,0}_4 & -3\boxed{1\,0}_4 & -\boxed{1\,0}_4 & +\boxed{1\,0}_4 \\ \vdots & \vdots & \vdots & \vdots \end{array} \tag{4.8}$$

or, more concisely in terms of the highest weights, as the branching rule

$$(1\,0)_0 > (1\,0)_0 - (1\,0)_1 - (1\,0)_3 + (1\,0)_4 + \ldots . \tag{4.9}$$

Example 3. $E_6^{(1)} > E_6^{(1)}$, $P = \mathrm{diag}(2, \ldots, 2|1)$
We consider the irreducible representation

$$\begin{pmatrix} & & 1 & & \\ & & 0 & & \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Application of P leads to a weight of level 2 and congruence class 0. There are only three dominant weights of this level and class:

$$\begin{pmatrix} & & 2 & & \\ & & 0 & & \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} & & 0 & & \\ & & 1 & & \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} & & 0 & & \\ & & 0 & & \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{4.10}$$

Proceeding with the counting of the multiplicities as before (using the tables of [10]), one gets the branching rule

$$\begin{pmatrix} & & 1 & & \\ & & 0 & & \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}_0 > \begin{pmatrix} & & 2 & & \\ & & 0 & & \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}_0 - \begin{pmatrix} & & 0 & & \\ & & 1 & & \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}_1$$
$$+6\begin{pmatrix} & & 2 & & \\ & & 0 & & \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}_1 + \begin{pmatrix} & & 0 & & \\ & & 0 & & \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix}_2$$
$$-7\begin{pmatrix} & & 0 & & \\ & & 1 & & \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}_2 + 22\begin{pmatrix} & & 2 & & \\ & & 0 & & \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}_2 + \ldots . \tag{4.11}$$

Example 4. $D_4^{(3)} > G_2^{(1)}$

The last example of subjoining branching rules refers to non-isomorphic algebras $D_4^{(3)} > G_2^{(1)}$. We consider the representation (1 0 0) of $D_4^{(3)}$ algebra. The multiplicities of the dominant weights on the highest 14 levels of the weight system are

$$\boxed{100}\ \boxed{100}\ \underset{2}{\boxed{100}}\ \underset{4}{\boxed{100}}\ \underset{6}{\boxed{100}}\ \underset{9}{\boxed{100}}\ \underset{16}{\boxed{100}}\ \underset{22}{\boxed{100}}\ \underset{33}{\boxed{100}}\ \underset{50}{\boxed{100}}\ \underset{70}{\boxed{100}}\ \underset{98}{\boxed{100}}\ \underset{143}{\boxed{100}}\ \underset{193}{\boxed{100}}\ \cdots \tag{4.12}$$

The projection matrix (2.9), being diagonal and having non-negative matrix elements, transforms dominant weights into dominant weights and non-dominant ones into non-dominant ones. Hence the projected system coincides with (4.12) except that now it refers to the subjoined algebra $G_2^{(1)}$. Using the known orbit multiplicities of $G_2^{(1)}$ representations, the $G_2^{(1)}$ orbits can be transformed to $G_2^{(1)}$ representations. The resulting branching rule from $D_4^{(3)}$ to $G_2^{(1)}$ is

$$(1\,0\,0)_0 > (1\,0\,0)_0 - (0\,0\,1)_1 + (1\,0\,0)_3 + 2(1\,0\,0)_6 - (0\,0\,1)_7 + (1\,0\,0)_9 - (0\,0\,1)_{10}$$
$$+2(1\,0\,0)_{12} - (0\,0\,1)_{13} + \ldots . \tag{4.13}$$

5. Final comment

Among the affine algebras there exist proper inclusions between isomorphic algebras. Those inclusions are closely related to a class of subjoinings described above in that they are the 'inverses' of these subjoinings. Moreover, maximality of one implies maximality of the other. The phenomenon occurs for untwisted and twisted Kac-Moody algebras.

We discussed subjoinings $G > G$ obtained by multiplying the depth of each weight by a positive integer k. This corresponds to replacing the imaginary root δ by δk^{-1}. The opposite operation (replacing δ by δk, or multiplying depths of weights by k^{-1} where k is a positive integer) defines an embedding $G \supset G$. This means retaining only those generators whose depths are multiples of k; they obviously close under commutation. As an example we consider the proper subalgebra $A_1^{(1)}$ in an isomorphic algebra denoted by the same symbol, i.e. the embedding $A_1^{(1)} \supset A_1^{(1)}$, with projection matrix $P = \mathrm{diag}(1, 1|\frac{1}{2})$ for the irreducible representation $(1\,0)_0$.

Following the now familiar prodecure used in section 4 we find

$$
\begin{array}{r c l l l l l l}
\boxed{10}_0 & & \boxed{10}_0 & & & & & \\
\boxed{10}_{\frac{1}{2}} & & & \boxed{10}_{\frac{1}{2}} & & & & \\
2\boxed{10}_1 & & \boxed{10}_1 & & +\boxed{10}_1 & & & \\
3\boxed{10}_{\frac{3}{2}} & = & & \boxed{10}_{\frac{3}{2}} & & +2\boxed{10}_{\frac{3}{2}} & & \\
5\boxed{10}_2 & & 2\boxed{10}_2 & & +\boxed{10}_2 & & +2\boxed{10}_2 & \\
7\boxed{10}_{\frac{5}{2}} & & & 2\boxed{10}_{\frac{5}{2}} & & +2\boxed{10}_{\frac{5}{2}} & & +3\boxed{10}_{\frac{5}{2}} \\
\vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{array}
\qquad (5.1)
$$

implying the branching rule

$$(1\,0)_0 \supset (1\,0)_0 + (1\,0)_{1/2} + (1\,0)_1 + 2(1\,0)_{3/2} + 2(1\,0)_2 + 3(1\,0)_{5/2} + \ldots.$$

Inverting the branching rule matrix

$$A_{n',n} = \begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 3 & \ldots \\ 0 & 1 & 1 & 1 & 2 & 2 & \ldots \\ 0 & 0 & 1 & 1 & 1 & 2 & \ldots \\ 0 & 0 & 0 & 1 & 1 & 1 & \ldots \\ 0 & 0 & 0 & 0 & 1 & 1 & \ldots \\ 0 & 0 & 0 & 0 & 0 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

defined in (5.1) by

$$(1\,0)_n \supset \bigoplus_{n'} (1\,0)_{n'} A_{n',n}$$

(of course the matrix has the property $A_{n'+a,n+a} = A_{n',n}$) yields the inverse branching rule

$$(1\,0)_0 > (1\,0)_0 - (1\,0)_1 - (1\,0)_3 + (1\,0)_4 - (1\,0)_5 \ldots$$

in agreement with (4.9).

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